Residual Spectrume under Isogeny

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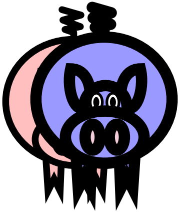
Pop quiz: Which of the following is NOT Jeju famous for?

1. 
2. 
3. L-functions.

- $\mathbf{G}$: a (split) connected reductive group over F, $\mathbf{Z}$: the center of $\mathbf{G}$,
- ω: a grössencharacter of F

$L^2(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}_F),\omega) = \{f : \mathbf{G}(\mathbb{A}) \to \mathbb{C} :$

$$\int_{\mathbf{G}(F)\mathbf{Z}(\mathbb{A})\backslash\mathbf{G}(\mathbb{A})} |f(g)|^2 dg < \infty$$

and

$$f(zg) = \omega(z)f(g), \quad z \in \mathbf{Z}(\mathbb{A}), \quad g \in \mathbf{G}(\mathbb{A})\}$$

with the right regular action of $\mathbf{G}(\mathbb{A})$.

- $\mathbf{B} = \mathbf{TU} \subset \mathbf{P} = \mathbf{MN} \subset \mathbf{G}$:Borel and parabolic subgroups of $\mathbf{G}$.
- $\mathfrak{a}$: the Lie algebra of the connected component of the center $\mathbf{A}$ of $\mathbf{M}$.
- Denote by

$$I(\lambda, \pi) = \mathrm{Ind}_M^G(\pi \otimes \exp(\langle \lambda, H_P(\cdot) \rangle))$$

the induced representation where π is an irreducible automorphic representation of $M = M(\mathbb{A})$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$.
- Let K be a maximal compact subgroup for which the Iwasawa decomposition holds

$$\mathbf{G}(\mathbb{A}) = K \cdot \mathbf{N}(\mathbb{A})\mathbf{M}(\mathbb{A}).$$

- Let $\mathcal{H}_{P,\pi,\sigma}$ be the space of functions

$$f : \mathbf{N}(\mathbb{A})\mathbf{M}(F)\mathbf{M}(\mathbb{R})^\circ \backslash \mathbf{G}(\mathbb{A}) \to \mathbb{C}$$

with certain conditions where σ is an irreducible representation of K.

- Set
$$I_{PW}(M,\pi) = \{\phi : \mathfrak{a}_{\mathbb{C}}^* \to \mathcal{H}_{P,\pi,\sigma} : (*)\}$$
and $(*)$ requires ϕ is of Paley-Wiener type and more.
- For $\phi \in I(M,\pi)$ and $\lambda \in \rho_{\mathbf{P}} + C^+$, set
$$\theta_\phi(g) = \left(\frac{1}{2\pi i}\right)^{\dim \mathbf{A}/\mathbf{Z}} \int_{\mathrm{Re}(\lambda)=\lambda_0} E(g,\phi(\lambda),\lambda)d\lambda$$
where $E(g,\phi(\lambda),\lambda)$ is the Eisenstein series.
- Define
$$L^2(G(F)\backslash \mathbf{G}(\mathbb{A}),\omega)_{(M,\pi)}$$
to be the space spanned by $\theta_\phi \in I_{PW}(M',\pi')$ for
$$(M',\pi') \sim (M,\pi) \iff wM = M', w\pi \cong \pi'.$$

The theory of Langlands says that

$$L^2(G(F)\backslash \mathbf{G}(\mathbb{A}),\omega) = L^2_{dis}(G(F)\backslash \mathbf{G}(\mathbb{A},\omega) \oplus L^2_{cont}(G(F)\backslash \mathbf{G}(\mathbb{A}),\omega)$$

where

- $L^2_{dis}(G(F)\backslash \mathbf{G}(\mathbb{A}),\omega) = \bigoplus_{(M,\pi)} L^2_{dis}(G(F)\backslash \mathbf{G}(\mathbb{A}),\omega)_{(M,\pi)}$
- $L^2_{cusp}(G(F)\backslash \mathbf{G}(\mathbb{A}),\omega) = \bigoplus_{(G,\pi)} L^2_{dis}(G(F)\backslash \mathbf{G}(\mathbb{A}),\omega)_{(G,\pi)}$
- and

$$L^2_{res}(G(F)\backslash \mathbf{G}(\mathbb{A}),\omega) = \bigoplus_{(M,\pi),M\neq G} L^2_{dis}(G(F)\backslash \mathbf{G}(\mathbb{A}),\omega)_{(M,\pi)}.$$

- An F-morphism $\phi : \mathbf{G} \to \mathbf{G}'$ is an isogeny if it is surjective and the kernel is finite.
- An isogeny is central if it induces an isomorphism of $\mathbf{U}_\alpha$ onto its image.
- (Example) $\mathrm{Spin}_N \to \mathrm{SO}_N$ is a central isogeny.
- Is it true

$$\mathbf{G} \twoheadrightarrow \mathbf{G}' \implies \mathbf{G}(\mathbb{A}) \twoheadrightarrow \mathbf{G}'(\mathbb{A})?$$

Proposition

Let $\phi : \mathbf{G} \to \mathbf{G}'$ be a central F-isogeny. Then $\phi(\mathbf{G}(\mathbb{A}))$ is cocompact in $\mathbf{G}'(\mathbb{A})$.

The proof uses Galois cohomology.

- Let $\mathbf{G} = \mathbf{G}^D \cdot \mathbf{S}$ be a reductive group and let

$$\phi : \mathbf{G}^D \times \mathbf{S} \to \mathbf{G}$$

be the corresponding central isogeny. E.g.,

$$\mathrm{SL}_n \times \mathrm{GL}_1 \to \mathrm{GL}_n.$$

- Notation: $*^D$ means the object associated with $\mathbf{G}^D$. For example, χ is a character of $\mathbf{T}(\mathbb{A}) \subset \mathbf{G}(\mathbb{A})$ and χ^D is a a character of $\mathbf{T}^D(\mathbb{A}) \subset \mathbf{G}^D(\mathbb{A})$.

Theorem

Let $\phi : \mathbf{G}^D \times \mathbf{S} \to \mathbf{G}$ be the central isogeny as before and assume $\dim \mathbf{S} = 1$. Given a unitary character χ of $\mathbf{T}(F)\backslash\mathbf{T}(\mathbb{A})$, let $\chi^D := \phi^*\chi|_{\mathbf{T}^D(\mathbb{A})}$ and let $\omega := \chi|_{\mathbf{S}(\mathbb{A})}$. Then ϕ induces an isomorphism

$$L^2_{dis}(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}),\omega)_{(T,\chi)} \approx L^2_{dis}(\mathbf{G}^D(F)\backslash\mathbf{G}^D(\mathbb{A}))_{(T^D,\chi^D)}.$$

Conversely, given an irreducible unitary character χ^D of $\mathbf{T}^D(F)\backslash\mathbf{T}^D(\mathbb{A})$ and a grössencharacter ω of F, there exists a unitary character χ of $\mathbf{T}(F)\backslash\mathbf{T}(\mathbb{A})$ such that $\chi^D \otimes \omega = \phi^*\chi$ and the isogeny induces

$$L^2_{dis}(\mathbf{G}^D(F)\backslash\mathbf{G}^D(\mathbb{A}))_{(T^D,\chi)} \approx L^2_{dis}(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}),\omega)_{(T,\chi)}.$$

Residual spectrum of GL_n supported on Borel subgroup.) Write a character of $\mathbf{T}(\mathbb{A}) \subset \mathrm{GL}_n(\mathbb{A})$ as

$$\chi = \chi(\mu_1, ..., \mu_n).$$

Moeglin and Waldspurger showed that

$$L^2_{dis}(\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}), \omega)_{(T,\chi)} \implies \chi = \chi(\mu, \cdots, \mu), \quad \mu^n = \omega$$

and that it is isomorphic to $\pi = \otimes_v \pi_v$ where

$$\pi_v = R_{\mathrm{GL}_n}(\lambda_B, \chi_v, w_0) I_{GL_n}(\lambda_B, \chi_v).$$

Residual spectrum of SL_n supported on Borel subgroup.) Let $\mathbf{G} = \mathrm{GL}_n$. Then $\mathbf{G}^D = \mathrm{SL}_n$ and there is an isogeny

$$\phi : \mathrm{SL}_n \times \mathrm{GL}_1 \to \mathrm{GL}_n.$$

Given χ^D (for $\mathrm{SL}_n = \mathbf{G}^D$) and a Grössencharacter ω, there is χ (for GL_n) such that $\chi^D \otimes \omega = \phi^* \chi$ and

$$L^2_{dis}(\mathrm{SL}_n(F)\backslash \mathrm{SL}_n(\mathbb{A}))_{(T^D,\chi^D)} \cong L^2_{dis}(\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}), \omega)_{(T,\chi)}$$

Then

$$\chi^D = \mathbf{1}_{T^D}, \quad \omega = \mathbf{1}_F$$

and $L^2_{dis}(\mathrm{SL}_n(F)\backslash \mathrm{SL}_n(\mathbb{A}))_{(T^D,\mathbf{1}_{T^D})}$ is spanned by $\pi^D = \otimes_v \pi_v^D$ where

$$\pi_v^D = R_{\mathrm{GL}_n}(\lambda_B^D, \mathbf{1}_{T^D}, w_0) I_{SL_n}(\lambda_{B^D}, \mathbf{1}_{T^D}).$$

Remark

The isogengy

$$\mathrm{Sp}_{2n} \times \mathrm{GL}_1 \to \mathrm{GSp}_{2n}$$

and the knowledge of residual spectrum of Sp_{2n} supported on Borel subgroup ((almost) determined by Henry Kim) determine that of GSp_{2n}.

Similarly

$$\mathrm{Spin}_{2n+1} \to \mathrm{SO}_{2n+1}$$

gives

Theorem

$L^2_{dis}(\mathrm{Spin}_{2n+1}(F)\backslash\mathrm{Spin}_{2n+1}(\mathbb{A}))_{(T,\chi)} = 0$ unless $\chi = \phi^*\chi'$ where

$$\chi' = \chi(\underbrace{\mu_1, ..., \mu_1}_{r_1}, \dots, \underbrace{\mu_k, ..., \mu_k}_{r_k})$$

where $\mu_1, ..., \mu_k$ are distinct non-trivial quadratic grössencharacters of F and $r_1 \geq \cdots \geq r_k \geq 1$, $r_1 + \cdots + r_k = n$. In such a case, we have

$$L^2_{dis}(\mathrm{Spin}_{2n+1}(F)\backslash\mathrm{Spin}_{2n+1}(\mathbb{A}), \omega)_{(T,\chi)} \approx L^2_{dis}(\mathrm{SO}_{2n+1}(F)\backslash\mathrm{SO}_{2n+1}(\mathbb{A}))_{(T',\chi')}.$$

Remark

1. There is a series of reductions

$$\text{reductive} \to \text{semisimple} \to \text{almost simple}$$

for consideration of residual spectrum supported on Borel subgroups.

2. More stories on Arthur parameter, $\mathrm{Res}_{E/F}\mathbf{G}$ etc.

Thank you!

Final Exam: How many times are L-functions mentioned in the talk?

1. Once
2. Twice
3. Many
4. None.